# Investigation of the free vibrational behavior of carbon nanotube-reinforced composite beams

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**Abstract:** This study investigates the free vibration behavior of nanocomposite beams reinforced with agglomerated single-walled carbon nanotubes (SWCNTs) embedded in an epoxy matrix. The effective material properties of the reinforced composite are estimated using the Weng model, and the equation of motion for the beam is derived based on Euler-Bernoulli beam theory. A MATLAB code is developed to determine the natural frequencies and eigenmodes of the nanocomposite beams. The results demonstrate that the elastic modulus of the agglomerated CNT composite increases with the increasing volume fraction of CNTs, leading to an increase in natural frequencies.

**Keywords:** Free Vibration, Beam, Carbon, Nanotubes, Composite, Single-Walled.

## 1. Introduction

A beam is a structural element that is capable of withstanding load primarily by resisting bending**.** The important elements of this technological structure appear in various forms, including various artifacts, such as structural elements of high-rise buildings, railways, long-span bridges, flexible satellites, gun barrels, robotic arms, aircraft wings [1]. Therefore, Marur have modeled composites of epoxy reinforced with spherical glass particles and Unit cell models are employed to model the composite. The three-unit cell: Cylindrical, Spherical, and cubical shape with spherical inclusion were taken to evaluate the effective elastic properties [2]. The study [3] finds the natural frequencies of vibration, which are rather high because of the great rigidity of the structure. Different approaches have been proposed for the estimation of mechanical properties of nanocomposites, such as molecular dynamic (MD) simulations, Mori–Tanaka (MT), Halpin–Tsai (HT), and the extended rule of mixture (EROM). The material is chosen according to specific applications and environmental loads. These materials have multiple advantages, which can make them attractive from the point of view of their application potential. It can be an improvement in rigidity, resistance to fatigue, resistance to corrosion or thermal conductivity in addition to having a gradation of properties thus making it possible to increase or modulate performances. Such as reducing local constraints or even improving heat transfer [4]. The Euler-Bernoulli beam theory, also referred to as Engineer's beam theory, Classical beam theory, or simply beam theory, is a simplified version of the linear theory of elasticity. The calculation of natural frequencies of continuous structures [5]; including the effects of geometric characteristics (length and cross-sectional area) and boundary conditions are obtained and discussed for the first four modes. Research carried out on epoxy composites reinforced with agglomerated CNTs gives interesting results and demonstrates that the use of long nanotubes can lead to better reinforcement. Weng model to estimate the effective material properties of CNT reinforced composites [6].

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This method can be used for different cases of nanocomposites reinforced with aligned or randomly oriented spheres of agglomerated CNTs Though, the traditional rule of mixture and extended rule of mixture do not handle such variety of factors. The main is to understand and predict their dynamic response under various conditions, this research holds significant importance due to the unique properties and potential applications of carbon nanotube-reinforced composite beams, this is a crucial step in unlocking their full potential for various engineering applications and advancing the field of nanotechnology

## 2. Nanocomposite beam reinforced by inclusions of CNTs

### *2.1 **Mori–Tanaka approach***

Due to their small diameter, low radial elastic modulus, and high aspect ratio, CNTs exhibit a tendency to agglomerate in polymer matrices. These characteristics contribute to reduced bending stiffness [7]. In this section to study the influence of the agglomeration of CNTs on the effective elastic moduli of CNT-reinforced composites. CNTs tend to cluster together in a matrix, resulting in a non-uniform distribution. Areas with higher CNT concentrations are referred to as "inclusions" and are considered to have distinct elastic properties from the surrounding matrix. The total volume $V_r$ of CNTs in the RVE $V$ can be divided into the following two parts:

$$V_r = V_r^{inclusio} + V_r^{m} \tag{1}$$

where $V_r^{inclusion}$ and $V_r^{m}$ denote the volumes of CNTs dispersed in the inclusions and in the matrix, respectively. Introduce two parameters $\zeta$ and $\xi$ to describe the agglomeration of CNTs:

$$\zeta = \frac{V_r^{inclusion}}{V_r}, \qquad \xi = \frac{V_{inclusion}}{V} \tag{2}$$

where $V_{inclusion}$ is the volume of the sphere inclusions in the RVE. $\xi$ , denotes the volume fraction of inclusions with respect to the total volume $V$ of the RVE When $\xi = 1$, nanotubes are uniformly dispersed in the matrix. The parameter $\zeta$ denotes the volume ratio of nanotubes that are dispersed in inclusions and the total volume of the nanotubes. When, $\zeta = 1$, all the nanotubes are located in the sphere areas. In the case where all nanotubes are dispersed uniformly, the average volume fraction $c_r$ of CNTs in the composite is:

$$c_r = \frac{V_r}{V} \tag{3}$$

The effective modulus of inclusions $E_{in}$ and their surrounding $E_{out}$ as

$$E_{in} = \frac{3}{8}\left\{\frac{c_{r(1-\zeta)}}{1-\xi}\ E_{CNT} + \left[1 - \frac{c_{r(1-\zeta)}}{1-\xi}\right] E_m\right\} + \frac{5}{8}\left\{\frac{(1-\xi)E_{CNT}E_m}{[(1-\xi) - c_r(1-\zeta)]E_{CNT} + c_r(1-\zeta)E_m}\right\} \tag{4}$$

$$E_{out} = \frac{3}{8\xi}\ [c_r\zeta E_{CNT} + (\xi - c_r\zeta)E_m] + \frac{5}{8}\ \left\{\frac{\xi E_{CNT}E_m}{(\xi - c_r\zeta)E_{CNT} + c_r\zeta E_m}\right\} \tag{5}$$

where both the matrix and the CNTs are considered to be isotropic, with Young's moduli $E_m$ and $E_{CNT}$, respectively.

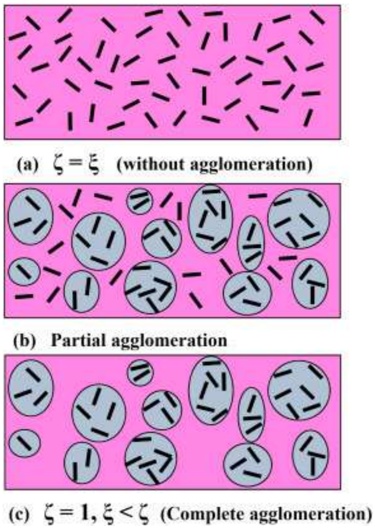



**Fig. 1 Complete agglomeration**

*2.2* ***Weng model***

Weng presented a two-phase model for the prediction of the effective elastic moduli of composites with aligned short fibers. The model is based on the Mori-Tanaka method and assumes that the composite consists of two phases: the matrix phase and the fiber phase. The matrix phase is assumed to be continuous and isotropic, while the fiber phase is assumed to be discontinous and anisotropic. For a 2-phase composite, the bulk and shear moduli reduce to:

$$\frac{K_{comp}}{K_m} = 1 + \frac{V_f}{\frac{3c_m K_m}{3K_m+4U_m} + \frac{K_m}{K_f-K_m}} \quad (6)$$

$$\frac{U_{comp}}{U_m} = 1 + \frac{V_f}{\frac{6}{5}\frac{c_m(K_m+2U_m)}{3K_m+4U_m} + \frac{U_m}{U_f-U_m}} \quad (7)$$

where the material properties of the constituents are calculated from the isotropic bulk and shear moduli of the matrix (phase m) and the inclusion (phase f). Moreover, following their notation, $c_m$ and $c_f$ are the volume fraction of the matrix and inclusion phase, respectively, with $c_m + c_f = 1$. From the above expressions, the Young's modulus, normalized by the Young's modulus of the matrix:

$$\frac{E_{comp}}{E_m} = \frac{K_{comp}U_{comp}(3K_m + U_m)}{3K_{comp}K_m + U_{comp}U_m} \quad (8)$$

where the bulk and shear moduli of each phase

$$K_m = {}^{E_m}\!/_{3(1-2\nu_m)} \; and \; U_m = {}^{E_m}\!/_{2(1+\nu_m)} \quad (9)$$

$$K_f = {}^{E_f}\!\Big/_{3(1-2\nu_f)} \; and \; U_f = {}^{f}\!\Big/_{2(1+\nu_f)} \quad (10)$$

## 3. Mathematical Formulation

Consider an elastic beam of length L, Young's modulus *E*, and mass density $\rho$ with uniform cross section *A*, as shown in Figure 1. This theory enables the calculation of load-carrying and deflection characteristics of beams.

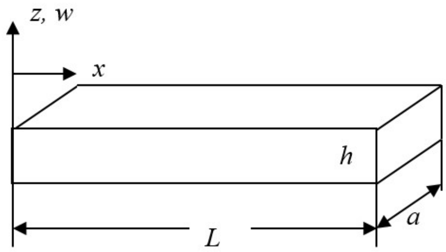



**Fig. 2 Geometry of the beam**

Using Euler-Bernoulli beam theory, one can obtain the equation of motion of a beam with homogeneous material properties and constant cross section as follows [12] where *I,* is the area moment of inertia of the beam cross section, *w* is the transverse displacement, and *t* is time.

$$\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad 0 \le x \le L \quad (13)$$

The solution of the Eq. (10) is sought by separation of variables. Assume that the displacement can be separated into two parts: one is depending on the position and the other is depending on time, as follows:

$$w(x,t) = X(x)T(t) \quad (14)$$

Substituting Eq. (11) into Eq. (10) and after some mathematical rearrangements, we obtained:

$$\frac{EI}{\rho A X(x)}\frac{\partial^4 X(x)}{\partial x^4} = \frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} \quad (15)$$

Each side equal to$(-\boldsymbol{\omega}^2)$ to have to have simple harmonic motion in the system:

$$\frac{EI}{\rho A X(x)}\frac{\partial^4 X(x)}{\partial x^4} = \frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -\omega^2 \quad (16)$$

The position variable is:

$$\frac{\partial^4 X(x)}{\partial x^4} - \delta^4 X(x) = 0 where \delta^4 = \omega^2 \frac{\rho A}{EI} \quad (17)$$

The time variable is:

$$\frac{\partial^2\ T(t)}{\partial t^2} + \omega^2 T(t) = 0 \quad (18)$$

Equation (5) solves as:

$$X(x) = C_1 \sinh(\delta x) + C_2 \cosh(\delta x) +$$

$$C_3 \sin(\delta x) + C_4 \cos(\delta x) \quad (19)$$

Equation (6) solved as:

$$T(t) = C_5 \sin(\omega t) + C_6 \cos(\omega t) \quad (20)$$

where, $C_1 , \cdots , C_5$ are constant. We multiplied (16) by (17) to get:

$$w(x,t) = (C_1 \ \sinh + C_2 \cosh(\delta x) + C_3 \sin(\delta x) + C_4 \ \cos(\delta x)) \times (C_5 \ \sin(\omega t) + C_6 \ \cos(\omega t)) \quad (21)$$

The constants $C_1 , C_2 , C_3$ and $C_4$ obtained from the boundary conditions, $C_5$ and $C_6$ obtained from the initial conditions. From equation (14) we get the natural frequency of the beam:

$$f_n = \frac{\omega}{2\pi} \quad (Hz) \quad (22)$$

### 3.1 *Boundary conditions*

The beam equation is a fourth-order differential equation, which means that it has four derivatives. This means that there are at most four conditions that need to be met in order to solve the equation. These conditions are usually, called boundary conditions, and they can model different things, such as supports, point loads, moments, or other effects here is a different boundary conditions.

### 3.2 *Material properties of CNTRCs*

A beam with Cross-sectional properties *h* = 0.01m Height of the beam, *b* = 0.02 m, width of the beam and length of *L* = 1 m. The effective material properties of CNTRCs are determined. Epoxy matrices reinforced with The CNT type used is the single-walled armchair, its tube thickness and diameter are respectively *th* = 0.35 nm and *d* = 1 nm, and the Young's modulus and Poisson's ratio. CNT: $E_{CNT}$= 1000 (GPa), $v_{cnt}$=0.3, $\rho_{CNT} = 2300 \ \mathrm{KG}/m^3$ . Matrix : $E_m$ =3.5 (GPa), $v_m$=0.3. $\rho_{CNT} = 1.2 \ \mathrm{KG}/m^3$[8].

**Table 1 Different boundary conditions**

| Beam configuration | Clamped-free | Clamped-clamped | Simply supported |
|---|---|---|---|
| at $x = 0$ | $w = 0$ and $\frac{dw}{dx} = 0.$ | $w = 0$ and $\frac{d^2w}{dx^2} = 0.$ | $w = 0$ and $\frac{d^2w}{dx^2} = 0.$ |
| at $x = 1$ | $M = \frac{d^2w}{dx^2} = 0$ and $V = \frac{d^3w}{dx^3} = 0$ | $w = 0$ and $\frac{dw}{dx} = 0.$ | $w = 0$ and $\frac{d^2w}{dx^2} = 0.$ |

## 4. Numerical Results and Discussion

The elastic moduli of the composite with Complete Agglomeration of CNTs: $\zeta$ =1 all the CNTs are located within the agglomerated inclusions. $\xi$ = 0.2, 0.4, 0.6, 0.8 to $\xi = 1$ all the CNTs are uniformly dispersed in the matrix, in this case $E_{out} = E_m = 3.5 \ GPA$ still the same $E_{in} at \ \xi = 0.2, 0.4, 0.6, 0.8, 1.$

### 4.1 *Elastic moduli of the composite*

The figure shows at $\mu$ =0.2 the Young's modulus has the higher increasing in function of the volume fraction, The decrease in $\mu$ with the increasing of $c_r$ Doesn't lead to a better mechanical performance (a decreasing in E the elasticity moduli due to the significant agglomeration effect,

The results show that the elastic modulus of the composite increases rapidly with $\xi = 0.2$ and $\xi = 0.8$. However, the agglomeration effect of the CNTs can start to weaken the composite and reduce its elastic modulus. Does not lead to a better mechanical performance.

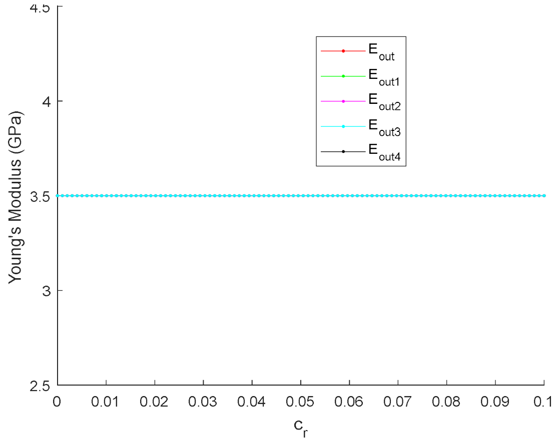



**Fig. 3 Young's Modulus of the surrounding matrix for Different Levels of Agglomeration and CNT Volume Fraction**

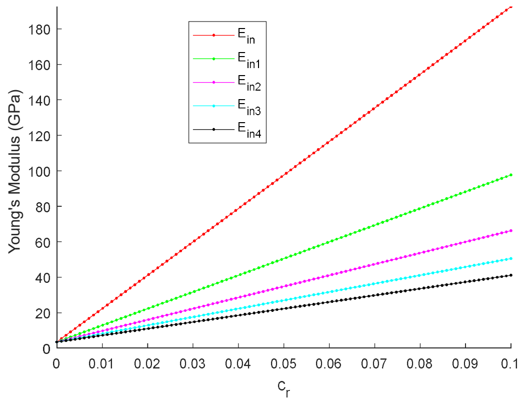
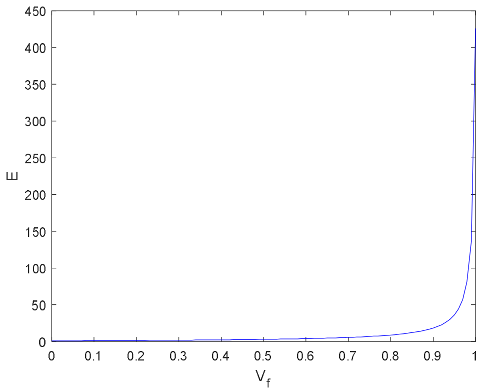
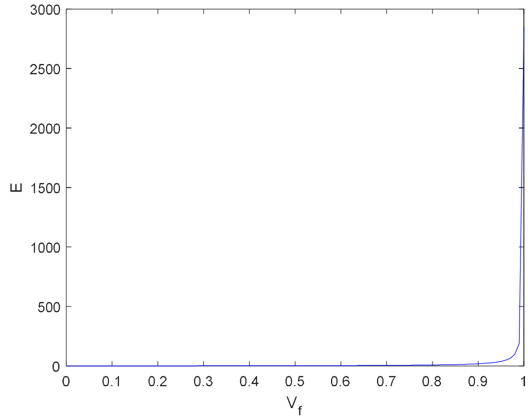



**Fig. 4 Young's Modulus for Different Levels of Agglomeration and CNT Volume Fraction**




(a)




(b)

**Fig. 5 Elastic moduli of the composite $c_r = 0.8$ (a) $\xi = 0.2$, (b) $\xi = 0.8$**

### *4.2 Natural frequencies*

Depending on various boundary conditions, the first few natural frequencies and the normalized mode shapes for the following beam configurations at different distributions were found out: clamped free, clamped-clamped and simply supported-simply supported mentioned in table 2.

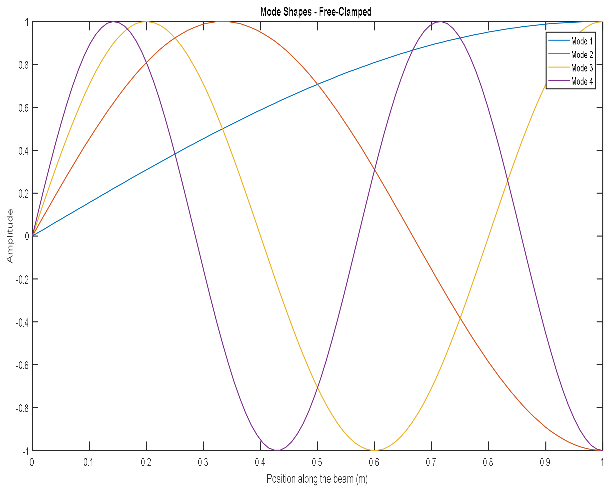
**Table 2 Natural frequencies at different distributions**

| | $f_1$ (Hz) | $f_2$ (Hz) | $f_3$ (Hz) | $f_4$ (Hz) |
|---|---|---|---|---|
| $\xi = 0.2$ | 8.91e-06 | 3.56e-05 | 8.02e-05 | 1.42e-04 |
| $\xi = 0.8$ | 8.86e-06 | 3.54e-05 | 7.98e-05 | 1.419e-04 |

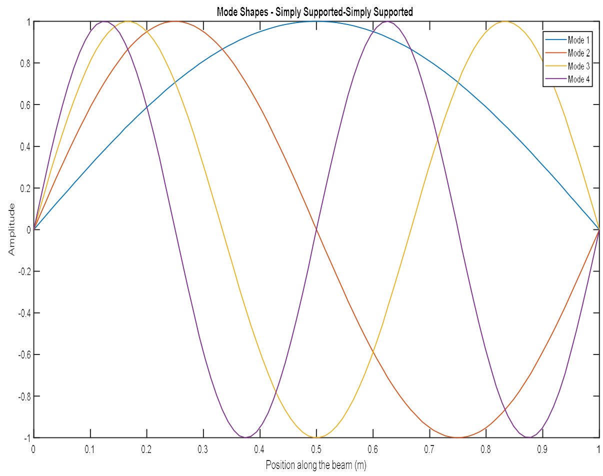
Agglomeration can adversely affect the natural frequencies of nanocomposite materials by diminishing the effective elastic modulus. This reduction in elastic modulus, in turn, leads to a decrease in natural frequencies.

### *4.3 Mode shapes*

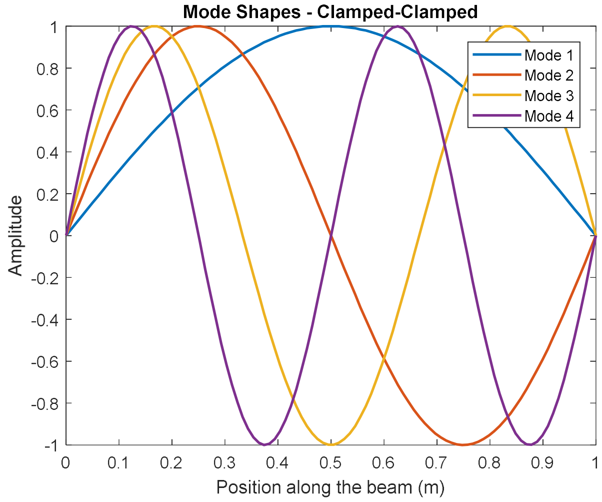
Mode shapes provide insight into how a structure deforms during vibration and are crucial in understanding its dynamic behavior. They are represented graphically and are a fundamental concept in structural dynamics, the first mode shapes of all kind of boundary conditions at $\xi = 0.2$ and $\xi = 0.8$ distribution that shows the figures .The mode shapes themselves are largely independent of the distribution of any additional materials, like agglomeration or nanotubes, and Mode shapes are primarily determined by the geometry and boundary conditions of the structure. The introduction of nanoparticles or nanotubes might alter the material properties, but as long as the overall geometry and boundary conditions remain the same, the fundamental mode shapes will remain consistent.




**(a)**




**(b)**




**(c)**

**Fig. 6 Mode shapes: (a) Clamped-free (b) Clamped-Clamped (c) Simply supported**

## 5. Conclusions

CNT agglomeration within composite materials can significantly reduce their natural frequencies. This decrease primarily stems from the reduction in stiffness caused by the agglomeration. Agglomerated nanotubes tend to weaken the elastic modulus, strength, and toughness of the composite. This weakening occurs because agglomerates act as defects within the material. These defects disrupt the efficient transfer of load between the nanotubes and the matrix, ultimately leading to premature failure. Consequently, minimizing CNT agglomeration during the fabrication of composite materials is crucial for optimizing their performance and achieving desired properties.

## References

[1] Khadri Y., Tekili S., Daya E. M., Daouadji A., Guenfoud M., M. Merzoug M.: Analysis of the Dynamic Response of Bridges Under Moving Loads. International Review of Mechanical Engineering 2(1), 91–99 (2009).

[2] Prabhakar R. Marur, "Estimation of effective elastic properties and interface stress concentrations in particulate composites by unit cell methods" Acta Materialia 52 (2004) 1263–1270.

[3] Odegard, G. M., Gates, T. S., Wise, K. E., Park, C., and Siochi, E. J.: Constitutive Modeling of Nanotube-

Reinforced Polymer Composites. Compos. Sci. Technol. 63(11), 1671–1687(2002).

[4] Zhang et al, A Study on the Effect of Nanotube Waviness and Agglomeration on the Mechanical Properties of Carbon Nanotube-Reinforced Composites, j*ournal of Nanomaterials,* (2022)

[5] Mura, T, Micromechanics of Defects in Solids, Martinus Nijh off Publishers, Dordrecht, ., 1987

[6] G. J. Weng, "Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions," International Journal of Enginering Science, vol. 22, no. 7, pp. 845-856, 1984.

[7] Shi D-L, Feng X-Q, Huang YY, et al. The effect of nanotube waviness and agglomeration on the elasticproperty of carbon nanotube reinforced composites J Eng Mater

[8] Technol 2004; 126: 250–257.B. Ramgopal Reddy, K. Ramji, “Modeling and Evaluation of Effective Elastic Properties of Carbon Nanotubes Reinforced Carbon Fiber/Epoxy Multiscale Composites" Materials Today: Proceedings 21 (2020) 1099–1103.