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An Element-Free Galerkin Analysis of Elasto-Plastic Fracture Problems

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Abstract: In this paper, we present a theoretical and numerical analysis of elasto-plastic problems based on the element-free Galerkin method (EFGM) and the numerical analysis. The study has been examined in planar stress analysis around the tip of a crack and in its opening mode of loading. In the EFGM, the implementation of the Moving Least Squares (MLS) approximation is used to obtain the approximation function and the transformation method is proposed to impose the essential boundary conditions. The discritized variational formulation for elasto-plastic materials obeying to the von Mises criterion is presented. To examine the validity of this technique, stress fields in a plate with a crack have been calculated.

Key words: Elasto-plasticity, numerical analysis, EFGM, MLS, transformation method.

1. Introduction

The use of FEM in incremental plasticity is a common practice but it has its own limitations. In the last decade Belytschko, Lu and Gu Ref. [1] introduced the EFG method to reduce some of the shortcomings of FEM in the solution of elastic field problems. The paper of Nayroles, Touzot and Villon Ref. [2] namely ‘‘Generalizing the FEM’’ was a close work prior to the former one and this work by itself seems to be inspired by another work which is in the area of moving least square (MLS) interpolants Ref. [3]. After introducing of the EFGM, this method has been used in a wide range of different subjects such as dynamic fracture Ref. [4, 5], crack growth Ref. [6, 7], elastic plates and shells Ref. [8], and non-elastic stress analysis Ref. [9].

The EFG Method has already been employed in elasto-plastic range by Barry and Saigal Ref. [9]. However, in their elasto-plastic endeavor, stress analysis around the crack tip has not been considered.

The existence of singularities such as cracks, demand special trends to ensure the convergence of the numerical method. Moreover, coincidence of non-linearity and singularity phenomena produce higher order difficulties for numerical solutions. It has to be mentioned that in the harsh nonlinear solution manner the value of most variables change in each level of iterative procedure.

2. Moving Least Squares Approximation

An excellent description of MLS is given by Lancaster and Salkauskas Ref. [3]. The MLS approximation $u^h(x)$ is defined in the domain Ω by

$$u^h(x)=\sum_{j=1}^{nb} p_j(x)a_j(x)=p^T(x)a(x) \qquad (1)$$

where $p(x)$ is the basis function, nb is the number of terms in the basis function, and the coefficients $a_j(x)$ are also functions of x, which are obtained at any point x by minimizing a weighted discrete L_2 norm of:

$$\bar{J}=\sum_{i=1}^{m} w(x-x_i)(p^T(x_i)a(x_i)-u_i)^2 \qquad (2)$$

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where u_i is the nodal value parameter of $u(x)$ at node x_i, and m is the number of nodes in the neighborhood of x for which the weight function $w_i(x) = w(x - x_i) \neq 0$. Many kinds of weight functions have been used in meshless methods. The quartic spline weight function is used in this paper,

$$w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & \text{for } 0 \leq r \leq 1 \\ 0 & \text{for } r > 1 \end{cases} \quad (3)$$

where $r = \|x_i - x\| / d_{\max}$ is the normalized radius and $d_{\max}$ is the size of influence domain of point x_i.

Using the stationary condition for $\overline{J}$ with respect to $a(x)$, we can solve $a(x)$. And then, substituting it into (1), we have

$$u^h(x) = \sum_{i=1}^{m} \phi_i(x) u_i \quad (4)$$

where the MLS shape function $\phi_i(x)$ is defined by

$$\phi_i(x) = \sum_{j=1}^{nb} p_j(x) (A^{-1}(x) B(x))_{ji} \quad (5)$$

in the above equation, the matrices $A(x)$ (moment matrix) and $B(x)$ are given by

$$A_{jk} = \sum_{i=1}^{m} B_{ij} p_k(x_i), \ \ B_{ij} = w_i(x) p_j(x_i) \quad (6)$$

The MLS shape functions given in (5) do not, in general, satisfy the Kronecker's delta property, i.e., $\phi_i(x_j) \neq \delta_{ij}$. In order to overcome this difficulty, we use the transformation method whose the transformation matrix Λ is formed by establishing the relationship between the nodal value $u_j^h(x_k) \equiv \widehat{u}_{jk}$ and the "generalized" displacement u_{ij} by

$$u_j^h(x) = \sum_{i=1}^{m} \phi_i(x) u_{ji} \quad (7)$$

$$u_{ji} = \sum_{i=1}^{m} \Lambda_{ik}^{-1} \widehat{u}_{jk} \quad (8)$$

where $\Lambda_{ik} = \phi_i(x_k)$; by substituting (8) into (7), one can obtain

$$u_j^h(x) = \sum_{i=1}^{m} \sum_{k=1}^{m} \phi_i(x) \Lambda_{ki}^{-1} \widehat{u}_{jk} \equiv \sum_{k=1}^{m} \overline{\phi}_i(x) \widehat{u}_{jk} \quad (9)$$

where

$$\overline{\phi}_k(x) = \sum_{i=1}^{m} \Lambda_{ki}^{-1} \phi_i(x) \quad (10)$$

Note that

$$\overline{\phi}_k(x_j) = \sum_{i=1}^{m} \Lambda_{ki}^{-1} \phi_i(x_j) = \sum_{i=1}^{m} \Lambda_{ik}^{-1} \Lambda_{kj} = \delta_{ij} \quad (11)$$

and u^h and δu^h satisfy the following boundary conditions:

$$\left. \begin{aligned} u_j^h(x_i) &= \sum_{j=1}^{m} \overline{\phi}_j(x_i) \widehat{u}_{ij} \\ \delta u_j^h(x_i) &= \sum_{j=1}^{m} \overline{\phi}_j(x_i) \delta \widehat{u}_{ij} \end{aligned} \right\} \ \forall i \in \eta_{\bar{u}_i} \quad (12)$$

where $\eta_{\bar{u}_i}$ denotes a set of particle numbers in which the associated particles are located on boundary Γ_u. From (11), we directly obtain

$$\widehat{u}_{ji} = \bar{u}_j(x_i), \ \ \delta \widehat{u}_{ji} = 0, \ \ \forall i \in \eta_{\bar{u}_i}. \quad (13)$$

3. Governing Equations

In the field of solid mechanics the equilibrium equation for a continuous media under small displacements is given as

$$div(\Delta\sigma) + \Delta\bar{f} = 0 \ \text{ in } \Omega \quad (14)$$

with essential and natural boundary conditions as follows

$$\Delta u = \Delta\bar{u} \ \text{ on } \Gamma_u \quad (15)$$

$$\Delta t(\Delta\sigma) = \Delta\sigma n = \Delta\bar{t} \ \text{ on } \Gamma_t \quad (16)$$

In these relations, $\Delta\sigma$ is the stress tensor, $\Delta\bar{f}$ is the body force vector, $\Delta\bar{u}$ is the displacement vector, $\Delta\bar{t}$ is the traction force and n is the outward unit normal vector to the boundary Γ.

The incremental elastoplastic constitutive equations:

$$\Delta\sigma = C^{ep} \Delta\varepsilon \quad (17)$$

where $\Delta\sigma$ is the Cauchy stress increment tensor, C^{ep} is called the elasto-plastic tangent constitutive matrix and $\Delta\varepsilon$ is the strain increment tensor can be decomposed into elastic and plastic parts:

$$\Delta\varepsilon = \Delta\varepsilon^{e} + \Delta\varepsilon^{p} \tag{18}$$

The elastic constitutive relations:

$$\Delta\sigma_{ij} = C^{e}_{ijkl}\Delta\varepsilon^{e}_{kl} \tag{19}$$

where C^{e}_{ijkl} denotes elastic modulus tensor.

In this work, according to the von Mises criteria. The yield function is written as

$$f(\sigma,\bar{\varepsilon}^{p}) = (\frac{3}{2}\bar{\sigma}_{ij}\bar{\sigma}_{ij})^{0.5} - \sigma_{Y}(\bar{\varepsilon}^{p}) \tag{20}$$

where $\bar{\sigma}_{ij}$ denotes deviatoric stress and σ_{Y} the yield stress. $\bar{\varepsilon}^{p}$ indicates equivalent (or effective) plastic strain, and its time rate is defined as

$$\dot{\bar{\varepsilon}}^{p} = (\frac{2}{3}\dot{\varepsilon}^{p}_{ij}\dot{\varepsilon}^{p}_{ij})^{0.5} \tag{21}$$

From the associative flow rule, plastic strain can be writ-ten as follows:

$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\partial f / \partial\sigma_{ij} \tag{22}$$

where λ denotes plastic multiplier (or the flow amplitude) and $\partial f / \partial\sigma_{ij}$ defines the plastic flow direction.

The expression of elasto-plastic tangent tensor C^{ep} can be written as (see Ref. [10])

$$C^{ep} = C^{e} - \left(C^{e}N\right)\left(C^{e}\frac{\partial f}{\partial\sigma}\right)^{T} \Bigg/ \left(\frac{\partial f}{\partial\sigma}^{T}C^{e}N\right) \tag{23}$$

where $N(\sigma)$ is the unit flow direction vector defined as follows:

$$N(\sigma) = \frac{\partial f}{\partial\sigma} \Bigg/ \left\|\frac{\partial f}{\partial\sigma}\right\| = \frac{1}{\sqrt{2J_{2}}}s_{ij} \tag{24}$$

where $J_{2} = s_{ij}s_{ij}/2$ is the second invariant of the deviatoric stress tensor s of σ.

For the variational formulation in terms of displacements, the terms which do not depend on the incremental field Δu disappear and the bifunctional is reduced to (more details can be seen in Ref. [11]):

$$\Delta\beta(\Delta u) = \int_{\Omega}\left[\Delta\varepsilon(u)^{T}C^{ep}\Delta\varepsilon(u) - \Delta\bar{f}\,\Delta u\right]d\Omega - \int_{\Gamma_{t}}\Delta\bar{t}\,\Delta u d\Gamma \tag{25}$$

Therefore, the kinematical variational principle becomes

$$\underset{\Delta u^{k}\ KA}{Inf}\ \Delta\beta(\Delta u^{k}) \tag{26}$$

where Δu^{k} is the displacement field kinematically admissible (*KA*).

4. Least Squares Discretization

The displacement and strain increment fields are expressed with respect to an unknown nodal displacement increment vector ΔU as (see Ref. [11]):

$$\Delta u(x) = \phi(x)\Delta U \quad \text{and} \quad \Delta\varepsilon = B(x)\Delta U \tag{27}$$

where $\phi(x)$ is the matrix of the shape functions, $B(x) = \nabla_{s}(\phi(x))$ and ∇_{s} is the symmetric gradient operator.

Let us introduce the generalized nodal force increment vector:

$$\Delta F = \int_{\Omega}\phi^{T}\Delta\bar{f}d\Omega + \int_{\Gamma_{t}}\phi^{T}\Delta\bar{t}d\Gamma \tag{28}$$

The discretized form of the (25) is then a set of non-linear equations:

$$\Delta\beta(\Delta U) = \int_{\Omega}B^{T}C^{ep}B\Delta U d\Omega - \Delta F \tag{29}$$

In EFGM a crack can rather be model more easily than other methods. Here the rule is to omit that part of the shape function of any node which is situated in other side of a crack line. In the region near to the crack tip this rule has some ambiguity. In this work we have decided to increase the number of nodes to cover discontinuity fault. It should also be mentioned that, some modification technique has been used to overcome this problem Ref. [12].

We can use *J*-integral to represent a numerical value for stress singularity. Generally, in elasto-plastic situations *J*-integral is used representative to show the magnitude of stress singularity in crack tip. *J*-integral is an integral over a special function of stress, which is defined as follows Ref. [13] if we consider a crack in the x_k direction:

$$J_k = \int_{\Gamma_c} (W n_k - t_j u_{j,k}) d\Gamma_c, \quad k = 1,2 \qquad (30)$$

where Γ_c is a generic contour surrounding the crack front (belonging to a plane orthogonal to the crack plane in a point on the crack front), $W = \sigma.\varepsilon/2$ is the strain energy density, $t_j = \sigma_{ij} n_i$ is the traction vector evaluated along the contour Γ_c, with normal unit outwards components n_j and finally u_j is the displacement vector.

5. Numerical Result

In this example, we considered is a rectangular plate with an edge crack of length a=4*mm* under a distributed load as shown in Fig. 1. The load is 1000*Pa*, the size of the plate is $L \times l$=52×20mm^2 and the other parameters are the yield stress $\sigma_Y = 210MPa$, Poisson's ratio $\nu = 0.25$, and Young's modulus $E = 2\times10^5 MPa$, (see Ref. [14]).

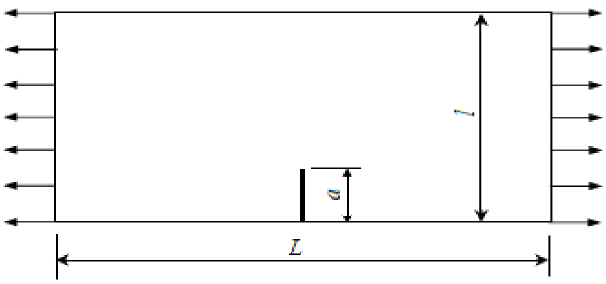

Fig. 1 Geometry and loading

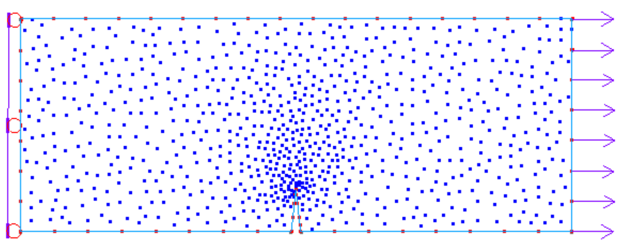

Fig. 2 Boundary conditions and irregular nodal arrangement

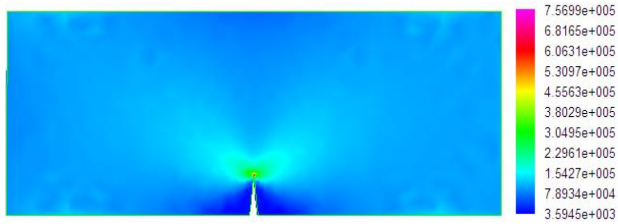

Fig. 3 Distribution of the von Mises Stress

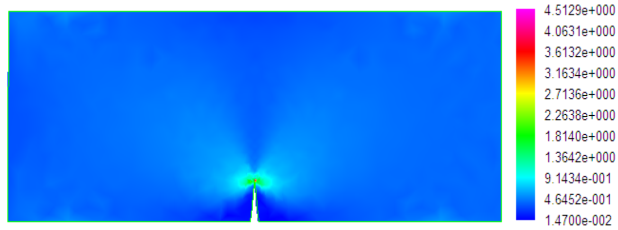

Fig. 4 Distribution of the equivalent strain

6. Conclusions

In this paper by combining EFG and incremental plasticity, a new solution method has been proposed. It is shown that the extension of EFGM to elasto-plastic stress analysis including the stress analysis in crack problems is feasible and that its results are reasonable. In addition, we used the transformation method to overcome the difficulty associated with the imposition of boundary conditions because the MLS shape functions, in general, didn't satisfy the Kronecker's delta property.

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